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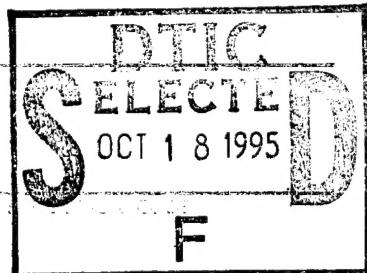
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13. ABSTRACT (Maximum 200 words)

The goals of this project were to (1) analyze numerical phenomena such as locking and boundary layers occurring in the modeling of elastic bodies, and obtain methods with robust performance, (2) extend this analysis to hierarchies of models, and (3) continue investigation into the p and h - p FEM. Specifically, the locking of hierarchy of plate models was analyzed to show that only the lowest order Reissner-Mindlin model effects were significant. Essentially locking-free h - mixed methods were established for the elasticity problem, Stokes flow, Reissner-Mindlin plate model and Naghdi shell. The h - p FE approximation of boundary layers was analyzed. Optimal convergence estimates for the 3-d version boundary element method were obtained. Numerical quadrature in the p version was analyzed and exponential convergence of an h - p quadrature scheme for singular integrals arising in boundary element and vortex methods was established. Wavelet based Galerkin boundary element methods as well as a convergent FEM for a class of nonconvex variational problems were developed.

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Hierarchical Modeling and Locking Effects in the Numerical Analysis of Multistructures (Grant F49620-92-J-0100)

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The goals of this project were to (1) analyze numerical phenomena such as *locking* and *boundary layers* occurring in the modeling of elastic bodies, and obtain methods with *robust* performance, (2) extend this analysis to *hierarchies* of models, and (3) continue investigation into the  $p$  and  $h - p$  FEM. Specifically, the locking of a hierarchy of plate models was analyzed to show that only the the lowest order Reissner-Mindlin model effects were significant. Essentially locking-free  $h - p$  *mixed* methods were established for the elasticity problem, Stokes flow, Reissner-Mindlin plate model and Naghdi shell. The  $h - p$  FE approximation of boundary layers was analyzed. *Optimal* convergence estimates for the 3-d  $p$  version boundary element method were obtained. Numerical quadrature in the  $p$  version was analyzed and exponential convergence of an  $h - p$  quadrature scheme for singular integrals arising in boundary element and vortex methods was established. Wavelet based Galerkin boundary element methods as well as a convergent FEM for a class of nonconvex variational problems were developed.

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## Introduction

Problems in stress analysis often involve structures that can be decomposed into canonical *thin* subdomains, such as (possibly laminated) beams, plates, shells, etc., linked by junctions. A *necessary condition* for good performance of the FEM on such structures is its robust performance on each thin component, considered separately.

Our primary objective has been to fully understand how to model the canonical subdomains and how to design FEM discretizations which are *robust* with respect to, for example, the thickness parameter. Such discretizations must avoid problems like *locking* and the existence of *boundary layers*. Our investigation has focussed on both these areas.

A second objective has been to investigate the feasibility of adaptive, hierarchical modeling of thin, three-dimensional domains. By this, we mean the automatic selection of a model from a hierarchy of two-dimensional models that converge to the actual solution as their order increases. (This is in contrast to the usual engineering technique of using a fixed model, such as the Reissner-Mindlin model.) Under this grant, we have developed *computable a-posteriori error estimators for the modeling error*. Existing  $p$  version FEM capabilities then provide a natural implementation of the hierarchical models. This work has applications in the use of hierarchical modeling for multistructures, where localized areas (such as joints) are replaced by models of higher order (or even the 3-d problem), while models of minimal complexity are used for other components.

The above ties in well with our continuing work on various aspects of the  $h - p$  version. In this regard, we have also investigated the  $p$  version of the boundary element method for three-dimensional problems, mixed  $h - p$  methods for the Stokes, elasticity, Reissner-Mindlin plate, and Naghdi shell problems, errors in the  $p$  version due to numerical quadrature, and  $h - p$  quadrature for singular integrals.

Moreover, we have proposed and analyzed a wavelet Galerkin scheme for the discretization of boundary integral equations of the first kind on polygonal domains that yields a “fast” algorithm with superconvergence properties for the solution at interior points. In addition, we have analyzed and implemented a finite element discretization for nonconvex variational problems.

### 1. Locking and Robustness of Standard FEMs

*Locking* is the phenomenon by which the numerical approximation of parameter-dependent problems deteriorates for values of the parameter close

to a limiting value.

The principal investigators M. Suri and C. Schwab, in joint work with Ivo Babuška, extended previous results for the analysis of locking phenomenon for the Reissner-Mindlin plate to a whole hierarchy of plate models. It was shown in [15] that there is no additional locking present for plate models that are of higher order than the Reissner-Mindlin plate. Hence, in the context of hierarchical modeling, the elements only have to be designed so that they properly deal with the locking of the lowest (Reissner-Mindlin) case.

Suppose the Reissner-Mindlin plate problem is approximated over a square domain with a uniform triangular or rectangular mesh, using the  $h$  version with polynomials of degree  $p$  for the rotations and degree  $q$  for the displacements. Then the locking and robustness orders in terms of  $N$ , the number of degrees of freedom, is summarized in Table 1, from [15].

Table 1: Locking and robustness for the  $h$  version with uniform meshes.

Type of Element	Degree $p$	Degree $q$	Order of locking, $r$ $f(N) = \mathcal{O}(N^r)$	Robustness order, $l$ $g(N) = \mathcal{O}(N^{-l})$
Triangle ( $\mathcal{P}_p$ )	1	$q \geq 1$	$r = 1/2$	$l = 0$
	$2 \leq p \leq 4$	$q = p$	$r = 1$	$l = (p - 2)/2$
	$p \geq 5$		$r = 1/2$	$l = (p - 1)/2$
	$2 \leq p \leq 3$	$l \geq p + 1$	$r = 1/2$	$l = (p - 1)/2$
Product ( $\mathcal{Q}_p$ )	$p \geq 4$	$l \geq p + 1$	$r = 0$	$l = p/2$
	1	$l \geq 1$	$r = 1/2$	$l = 0$
Trunk ( $\mathcal{Q}'_p$ )	$p \geq 2$	$q \geq p$	$r = 1/2$	$l = (p - 1)/2$
	1	$q \geq 1$	$r = 1/2$	$l = 0$
	2	$q = 2, 3$	$r = 1$	$l = 0$
		$q \geq 4$	$r = 1/2$	$l = 1/2$
	$p \geq 3$	$q = p$	$r = 3/2$	$l = (p - 3)/2$
		$q \geq p + 1$	$r = 1$	$l = (p - 2)/2$

In a second work on locking [18], an earlier mathematical definition of locking was developed into precise computable ways to quantify it. The difference between  $h$  and  $p/h - p$  techniques in combating locking was brought out using various computational results for problems of engineering interest such as nearly incompressible elasticity, anisotropic heat flow, and problems over thin domains. Various issues, such as the difference in locking when

different quantities of interest were calculated, the increase in locking when curved elements were used, and the superiority of high-order methods in combating locking were discussed in a computational framework.

## 2. $h - p$ Approximation of Boundary Layers

Boundary layers are an important phenomenon in the solution of plate and shell problems. Their numerical approximation has up till now been treated in an ad hoc fashion, usually by (uniform) over-refinement near the edges. In [19], we carried out a comprehensive investigation on how  $h$ ,  $p$  and  $h - p$  refinements could be best designed to *uniformly* capture such functions, for any thickness  $d$ . Our work is particularly useful in the context of  $h - p$  version codes, such as MSC PROBE and STRESSCHECK.

Since the boundary layer behavior manifests itself only in the direction perpendicular to the edge, a crucial aspect was to investigate the uniform approximation of 1-d boundary layer functions. A key result that we established is that the  $h - p$  version gives uniform exponential convergence if the first element is of size  $O(pd)$ , with only minimal additional elements required to mesh the remaining domain. Figures 1 and 2 illustrate this through numerical experiments on a singularly perturbed second-order elliptic problem on  $[-1,1]$  with exact solution

$$u_d(x) = 1 - \cosh(x/d)/\cosh(1/d),$$

which has boundary layers at both 1 and -1. It is seen that for different values of  $d$ , the 3-element mesh outperforms both the  $p$  version and the optimal  $h$  version mesh (which we showed must be *exponentially* graded) (it also outperforms the  $h - p$  version with more elements).

The paper [8] was presented at the AMS Mathematics of Computation conference at the University of British Columbia, Canada, August 9-13 by M. Suri. In it, we showed how two-dimensional elements that are both free of locking *and* uniformly approximate the boundary layer may be developed for the Reissner-Mindlin plate. This work has been extended in the Ph.D. thesis of Christos Xenophontos, which is currently under preparation.

The paper [20] was presented at the ICOSAHOM meeting in Houston, TX, June 5-9. In it, we extended the work in [19] to a model one-dimensional advection-diffusion problem, showing that a suitable variational formulation with the  $h - p$  version once again leads to exponential convergence.

The technical brief [21] discusses mesh design for structural plates. We showed how the improper mesh design in programs such as STRESSCHECK

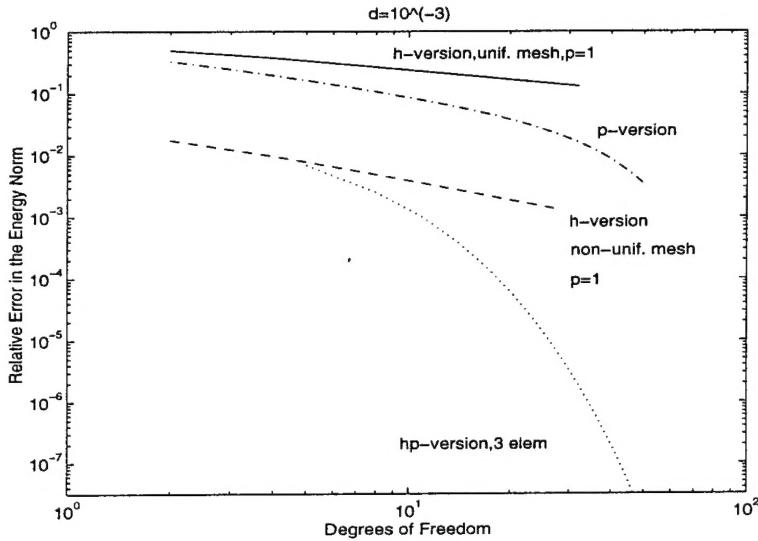


Figure 1: Comparison of various methods,  $d = 10^{-3}$

can lead to completely incorrect predicted shear forces, due to unsatisfactory resolution of the boundary layer. Following our theory in [19], the mesh should be refined near the edges, a strategy that has been explicitly adopted in the STRESSCHECK manual.

### 3. Mixed and Reduced-Constraint $h - p$ Methods

Although the  $p$  version is generally free of locking, various commercial  $h - p$  codes now offer the user a choice of both  $h$  and  $p$  refinement, the idea being that mesh refinement with low  $p$  is combined with low refinement and high  $p$ , depending on the nature of the subdomain. It is therefore essential that the method used be free of locking in areas of low  $p$  as well, i.e., where the convergence is achieved primarily by mesh-refinement ( $h$  version). This suggests a modification of the underlying variational method, to reduce the effect of the locking constraint. Such a modified variational form is often more accurate in recovering quantities of interest such as the pressure and the stress as well.

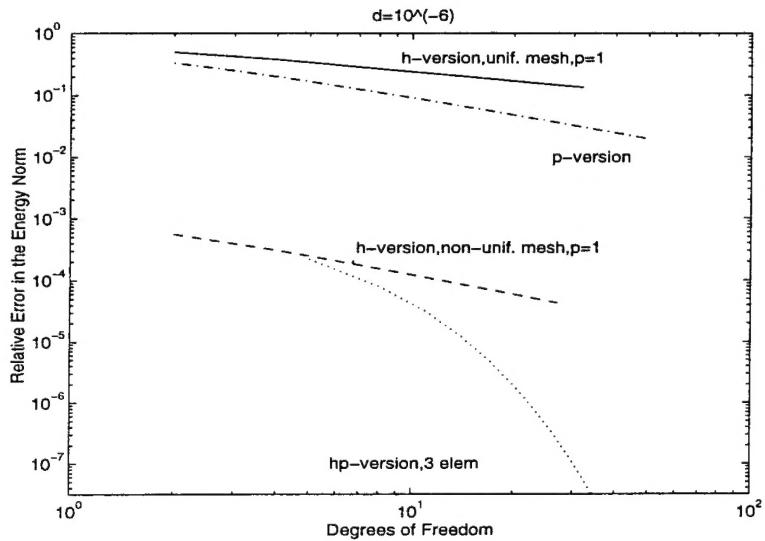


Figure 2: Comparison of various methods,  $d = 10^{-6}$

### (a) Elasticity and Stokes Problem

In joint work with R. Stenberg [12], we investigated the mixed  $h - p$  version for the problem of nearly incompressible elasticity, and its limit, the Stokes flow problem. The mixed  $h - p$  version combines the power of *both* high-order *and* mixed methods to combat locking. Unlike the standard method, where only locking-free approximations in the displacements (or velocities) can be guaranteed, the mixed  $h - p$  method can be designed to control the locking in the stresses (or pressures) as well.

We analyzed several families of rectangular mixed elements (including those in codes such as “NEKTON” and “P3CFD”), which had previously been analyzed either in terms of only the  $h$  version or only the  $p$  version, but not both. We showed that for the popular spectral element, “ $Q_N - Q_{N-2}$ ,” the  $h$ -approximability for the pressures is not optimally balanced against that for the velocities. We formulated a new mixed  $h - p$  element in which this balance is optimal, and which is, moreover, *minimal* in the number of degrees of freedom used, while still being locking-free for both velocities and pressures. Our elements have been implemented in a fully adaptive  $h - p$  method for fluid flow by J.T. Oden’s group at TICAM, with excellent results.

### (b) Reissner-Mindlin Plate

In [14], we extended our results above to obtain locking-free reduced constraint  $h - p$  elements for the Reissner-Mindlin plate. Our elements are variants of the well-known MITC elements, which had previously only been shown to be  $h$  stable.

### (c) Naghdi Shell

In [13], we formulated a reduced-constraint  $h - p$  finite element method on rectangular elements for the Naghdi shell model. This model suffers from locking in the bending-dominated case. We showed that by altering the underlying variational form, a method that was substantially free of locking both in terms of  $h$  and  $p$  was obtained.

### (d) Design Principles for Reducing Plate Constraints

Joint work with J. Pitkäranta [22] was motivated by the locking constraints found in shell problems. So far, satisfactory mixed method techniques to handle such constraints have not been developed. A natural strategy would be to extend the derivation of plate elements to the shell setting. However, mixed methods for the Reissner-Mindlin plate are derived by appealing to a Helmholtz decomposition, an indirect technique without an apparent analog for shells.

In our work, we developed a new means of deriving mixed elements for the Reissner-Mindlin plate, which *does* extend to shells. For the first time, the reduction operators and polynomial spaces used to treat the constraints are *directly* derived. Our analysis gives minimal conditions on plate elements needed to prevent locking, and also to approximate the boundary layers present. Using our analysis, a clear picture emerges for the comparison of currently available low-order ( $h$  version) plate elements, such as the MITC and Arnold-Falk ones.

We have now started applying our method to the problem of shell locking.

## 4. Hierarchic Modeling

In the area of hierarchic modeling of thin structures our principal objective has been the development of *computable, a-posteriori estimators* of the *modeling error*, i.e. the error between the solution of the three-dimensional problem and that of any model in the hierarchy.

### (a) Heat Conduction in a Thin Plate

C. Schwab has, in joint work with I. Babuška, completed an analysis of hierarchical models for the heat conduction problem in a thin plate [5]. There, a-posteriori estimators for the modeling error were obtained and their *asymptotic* (i.e. as the thickness of the structure tends to zero) and *spectral* (i.e. as the spectral order of the model tends to infinity) exactness were proved. Further, *computable a-posteriori bounds on the effectivity indices* for these estimators were obtained for the first time. Using symbolic manipulation, all bounds in our theory were shown to be sharp. These bounds allow the assessment of the reliability of the modeling error estimators for each given boundary value problem based on the computed solution. Moreover, these estimators, based on the residual tractions on the faces of the plate, were shown to be *locally asymptotically exact*, i.e. their local size gives a reliable indication of the subdomains in which the plate model needs improvement. This allows one to *adaptively* select different model orders in different parts of the domain.

The computational feasibility and efficiency of this strategy has been demonstrated in the paper [6], where it was shown that the estimators indicate reliably the boundary and interior layers present in the three dimensional problem, and that the local increase of the model order is computationally feasible and leads to near optimal reduction of the modeling error. It was found essential for the reliability of these modeling error estimators that an accurate numerical solution of the currently adopted two dimensional model be available, underlining once again the necessity of investigating the locking and boundary layer phenomena for the whole hierarchy of the models in a unified fashion.

### (b) Isotropic Elasticity and Monoclinic Plates

The above results were extended to hierachic plate models of uniform model order in the context of homogeneous, isotropic elasticity in the paper [4] and later to general, monoclinic plates in [24]. Here, once again, estimators that are guaranteed upper estimators can be obtained. The constants characterizing the estimators depend on the elastic moduli. We provided two ways for computing them:

- a) we gave analytic expressions for the constants by direct asymptotic analysis in [24],
- b) we showed how to compute them *numerically* by solving (once and for all) one generalized eigenvalue problem on the unit cube.

In the paper [23] we also proved the *asymptotic and spectral exactness*

of the modeling error estimators with the constants obtained according to method a). We expect the approach b) to be more general than a) and to apply also for variable thickness plates and shells.

Below is an important result for homogeneous and isotropic plates  $\Omega = \omega \times (-t/2, t/2)$  of thickness  $t$  with polynomial body force. It shows how the membrane and bending modeling error in the energy norm may be estimated by a computable estimator. Here,  $\mathbf{n} = (n_1, n_2, n_3)$  is the model order, denoting the polynomial degrees  $n_1, n_2, n_3$  chosen in the three displacement directions respectively.

$$\begin{aligned}\|\vec{u} - \vec{u}^{\mathbf{n}}\|_{E(\Omega)}^2 &= \|\vec{u}_{mem} - \vec{u}_{mem}^{\mathbf{n}}\|_{E(\Omega)}^2 + \|\vec{u}_{ben} - \vec{u}_{ben}^{\mathbf{n}}\|_{E(\Omega)}^2 \\ &\leq \frac{t}{2\mu} \left\{ \frac{\|r_{mem}^{\mathbf{n}}\|_{L^2(\omega)}^2}{\Lambda_{mem}} + \frac{\|r_{ben}^{\mathbf{n}}\|_{L^2(\omega)}^2}{\Lambda_{ben}} \right\}\end{aligned}\quad (1)$$

Here  $\mu$  is an elastic modulus and the constants  $\Lambda_{ben}$ ,  $\Lambda_{mem}$  depend only on the model order  $\mathbf{n}$  and the Poisson Ratio. They must be computed by solving (once and for all) a generalized eigenvalue problem on the unit cube in  $\mathbb{R}^3$ . The  $r^{\mathbf{n}}$  are the computable residual tractions on the faces of the plate.

To illustrate the performance of (1), we modeled (using PEGASYS) the bending of a clamped steel plate with a central hole due to normal loading (isotropic material,  $E = 10^7$ ,  $\nu = 0.3$ , thickness  $t = 0.1a$ ,  $a$ =edge length). To compute the solution of the  $\mathbf{n}$  models, a very simple mesh in was used with thin elements near the edge included for boundary layer resolution. The modeling error estimate (1) was compared to the true one (measured against a  $3-d$ ,  $h-p$ -FEM solution). Table 1 shows estimated and true modeling errors and the effectivity indices  $\Theta = \text{Estimator}/\|\vec{u}^{\mathbf{n}} - \vec{u}\|_{E(\Omega)}$  for models up to order 6. We see that (1) always overestimates the modeling

$\mathbf{n}$	(1, 1, 2)	(3, 3, 2)	(3, 3, 4)	(5, 5, 4)	(5, 5, 6)
$\Lambda$	2.49	4.49	4.49	6.49	6.49
$t\ r_{ben}^{\mathbf{n}}\ _{L^2(\omega)}^2/(2\mu\Lambda_{ben})$	2.333e-8	1.154e-8	6.557e-9	4.992e-9	3.624e-9
$\ \vec{u}_{ben}^{\mathbf{n}} - \vec{u}_{ben}\ ^2$	1.404e-8	4.271e-9	2.824e-9	2.448e-9	2.168e-9
$\Theta$	1.288	1.667	1.523	1.427	1.292

Table 2: Performance of the residual modeling error estimators for various model orders.

error with quite moderate effectivity indices and that (1) is accurate even for higher model orders. The theory has been generalized in the meantime to hierachic plate models with a) orthotropic and b) laminated materials, which are available in PEGASYS. These will be presented in a monograph which is presently under completion.

**(c) Fourier Analysis Techniques**

In [2], we developed the tool of *Fourier analysis* of the modeling error. This turned out to be quite useful in the analysis of locking phenomena in [15], and has been extended to include arbitrary model orders. The theory has been generalized in the meantime to hierachic plate models with general monoclinic materials ([24]) which are available in PEGASYS.

**(d) Subspace correction methods**

In [17], we started work on the convergence analysis of subspace correction methods for the iterative solution of hierachic plate models. There the connection between the selection of the transverse shape functions in the hierachic plate models and the convergence rate of certain block iterative procedures was analyzed, again with the tool of Fourier analysis developed in [2]. It was shown that a simple orthogonalization process applied to the transverse shape functions will yield iteration procedures with a convergence rate which *improves* as the plate gets thinner. In recent related work by a group in the U.K., it is actually shown that this selection of basis functions also ensures a convergence rate *independent* of the model order, i.e. we have a spectrally uniform convergence rate. For the near future (1995/6), extension of these results of elasticity problems is anticipated.

**(e) Boundary layers in hierachic models**

In [16], we analyzed in detail the boundary layer structure in hierachic models of homogeneous, isotropic plates. This was done to obtain insight into the discretization of the plate models themselves in connection with our  $h - p$  analysis in [19].

**5. The  $p$  Version Boundary Element Method**

The BEM is increasingly being used for approximating partial differential equations over polyhedral domains. *Optimal* rates of convergence for the  $p$  version BEM for the singularities in the case of three-dimensional problems were established in the paper [25]. In this work, we described the three

types of singular components that arise in the solution: edge, vertex and edge-vertex singularities, and analyzed the approximation of the traces of each of these by polynomial subspaces on the boundary. Asymptotic rates of convergence as  $p \rightarrow \infty$  that were optimal in the  $H^t$  norm over the boundary for  $0 \leq t \leq 1$  were established and applied to a model Neumann problem.

## 6. Quadrature Results

### (a) $h - p$ quadrature of singular and near singular integrals

In [7], C. Schwab applied ideas from  $h - p$  finite element methods to numerical evaluation of singular and near singular integrals which arise in many engineering applications (BEM, vortex methods, etc.). *Exponential convergence* with respect to the number of quadrature points was shown to hold uniformly for integrands in the countably normed space  $\mathcal{B}_\beta(\Omega)$ , containing in particular all weakly and near singular integrals arising in  $3 - d$  BEM.

### (b) Error in $p$ version in presence of quadrature

The question of which quadrature rule to use is an extremely important one in practice, since in commercial programs, the major portion of time spent is on numerically calculating the stiffness matrices (especially in three dimensions). Various topics related to numerical quadrature for the  $p$  version were investigated by Chang Kim as part of his Ph.D. thesis (under M. Suri), completed in Spring, 1992. In particular, the effect of using curved elements, unsmooth data, and the quadrature error in lower order norms were investigated theoretically and computationally. The results were published in [1].

## 7. Wavelet Galerkin Discretization of First Kind Boundary Integral Equations on Polygons

C. Schwab has, in joint work [3] with T. von Petersdorff, shown mathematically and computationally that an optimal order convergent discretization of *first kind boundary integral equations* on arbitrary polygons in the plane is possible in  $O(N \log N)$  memory and  $O(N(\log N)^2)$  operations where  $N$  denotes the *number of degrees of freedom on the boundary*. It has been shown that a) the corresponding stiffness matrices have bounded condition numbers in wavelet bases and b) that instead of  $N^2$  nonzero entries only  $O(N \log N)$  nonzero entries need be computed and stored while the optimal asymptotic convergence rate is preserved, even in the most negative norm on the boundary. The results cover interior and exterior boundary value problems for the Poisson, Helmholtz, plane elasticity and Stokes equations

alike. Spline and biorthogonal wavelets similar to those of Chui and Wang are used. The results have been extended to closed surfaces in  $R^3$  in [23].

## 8. FEM for Non-convex Variational Problems

The FEM discretization of an abstract minimization problem,  $\min F(u)$ , related to Newton's problem of minimal resistance of a body moving through a fluid was analyzed in [10]. The functional  $F$  was neither convex nor growing at  $\infty$ . It was shown that with polygonal domains and linear Courant triangles, the finite element approximations over appropriate admissible classes converged to a minimizer. A class of projected Newton methods for the discrete problems yielded locally superlinear convergence. Numerical experiments were performed for a model functional  $F$ .

## 9. Expository Work on $p$ and $h - p$ Versions

A joint paper written by I. Babuška and M. Suri [9] on the foundations of these methods was published in *SIAM Review*. The paper contains simplified proofs and numerous illustrations that bring out the main theoretical and numerical characteristics of these methods, and serves as an introduction for both mathematicians and engineers.

## 10. Benefits to USAF and Technology Transfer

A number of our results have been or will be implemented in commercial  $h - p$  codes used (among others) in the aerospace industry.

### (a) $h - p$ approximation of boundary layers.

The mesh-degree combinations suggested by us to resolve boundary layers were used to show how the  $h - p$  program STRESSCHECK (developed by Prof. B. Szabo at ESRD, Inc, St. Louis, Mo.) can extract quantities of interest such as stresses with great accuracy even in the presence of boundary layers. ESRD has issued a technical brief based on this work [21].

### (b) Modeling error estimators.

Our results in the papers [4] - [6] were discussed and used by ESRD Software, Inc. in St. Louis, MO in their development of the FE code PEGASYS which features advanced modeling capabilities for (laminated) plates and shells. It was demonstrated in ["SBIR Phase I final report: Hierarchic Modeling of laminated plates and shells", R. Actis, ESRD Software Inc., St. Louis, Mo] that the estimators proposed in [4] can be easily implemented and that the

layers and singularities require enhanced resolution by a refined model. The capability of variable model order and its self-adaptive selection analyzed in [11] will be implemented and also generalized to nonlinear problems in this commercial environment during the Phase II of the SBIR project.

(c) **Mixed  $h - p$  methods.**

The  $h - p$  elements developed by us for Stokes problem were the ones chosen for a fully adaptive implementation by A. Patra and J. T. Oden at TICAM.

(d) **Modeling of multistructures.**

We are working closely with MacNeal-Schwendler on this project. The idea is to be able to model substructures independently and then join them together using an interface technique. The end-product will be an  $h - p$  commercial code that has substructuring capability.

(e) **Numerical quadrature for singular BEM integrals.**

Subroutines based on the results on numerical quadrature have been incorporated into an industrial, 3-d boundary element code called ‘POLOPT’ of ABB (Asea Brown Boveri) Research division, Heidelberg, Germany, for electrostatics simulations.

(f) **USAF Ph.D. student.**

Major Lawrence Chilton of the USAF is doing his Ph. D. with us. He will serve on the mathematics faculty at Wright-Patterson AFB, Ohio upon completion.

### **Participating Professionals**

In addition to the PIs, the following personnel were involved in the research:

(a) Graduate (Ph.D.) Students

Chang Kim

Christos Xenophontos

Major Lawrence Chilton, USAF

Padmanabhan Seshaiyer

(b) Visiting Mathematicians and Consultants

Rolf Stenberg, Helsinki Institute of Technology

(c) Other Collaborating Mathematicians  
Ivo Babuška, University of Maryland College Park  
Juhani Pitkäranta, Helsinki University of Technology  
Tobias v. Petersdorff, University of Maryland College Park  
Bernd Kawohl, Universitaet Cologne, Germany

#### **Advanced Degrees Awarded**

Chang Kim, Ph.D, UMBC, 1992. Thesis title: "The effect of quadrature error in the  $p$  version of the finite element method." (The results are available in reference [1].)

#### **Seminars and Conference Talks**

- Jun '92** ICOSAHOM '92 (International Conference on Spectral and Higher Order Methods), Montpellier, France (June 21-25, 1992), "The  $p$  version of the boundary element method over polyhedral domains" (M. Suri) (organizer of minisymposium, "The  $p$  version of the finite element method")
- Nov '92** Brunel University, Uxbridge, U.K., "Locking and robustness in the finite element approximation of elasticity problems" (M. Suri)
- Jan '93** Oxford University, Oxford, U.K., "Locking and robustness in the finite element approximation of elasticity problems" (M. Suri)
- Apr '93** MAFELAP 7 (Seventh International Conference on the Mathematics of Finite Elements and their Applications), Uxbridge, U.K. (April 27-30, 1993), "The numerical resolution of boundary layers and locking effects in the Reissner-Mindlin plate model" (M. Suri) (PIs co-organized minisymposium, "The finite element approximation of plates and shells")
- May '93** Helsinki University of Technology, Espoo, Finland, "Locking effects in the finite element approximation of plate models" (M. Suri)
- Jun '93** University of Jyväskylä, Jyväskylä, Finland, "The  $p$  and  $h-p$  versions of the finite element method" (M. Suri)

- Aug '93** Symposium on Mathematics of Computation 1943-1993: A Half-Century of Computational Mathematics, University of British Columbia, Vancouver, Canada, August 9-13, 1993, "Locking and boundary layer effects in plate models" (M. Suri)
- Aug '93** Second U.S. National Congress on Computational Mechanics, Washington, D.C. (August 16-18, 1993), "Locking effects in the finite element solution of plate models" (M. Suri)
- Aug '93** "The optimal convergence rate of the  $p$ -version BEM on polyhedra", IABEM '93 conference at Braunschweig, FRG. (C. Schwab)
- Nov '93** "A-posteriori error estimation for adaptive, hierachic plate models" (C. Schwab), invited talk at the Biannual conference on computational mechanics at the University of Hannover (E. Stein, Chair)
- Mar '94** Georgia Institute of Technology, Atlanta, Georgia, "The mixed  $h - p$  method for problems in elasticity and Stokes flow" (M. Suri)
- Mar '94** Brown University, Providence, Rhode Island, "The mixed  $h - p$  method for problems in elasticity and Stokes flow" (M. Suri)
- Jul, '94** 1994 SIAM Annual Meeting, San Diego, CA (July 25-29, 1994), "Mixed  $h - p$  finite elements for elasticity problems" (M. Suri)
- Oct '94** Boundary Element Methods: Applications and Error Analysis, Oberwolfach, Germany (October 2-8, 1994), "The optimal  $p$  version approximation of singularities on polyhedra in the BEM" (M. Suri)
- Oct '94** Boundary Element Methods: Applications and Error Analysis, Oberwolfach, Germany (October 2-8, 1994), "Wavelet Methods for Boundary Integral Equations" (C. Schwab)
- Mar '95** Texas A & M University, College Station, TX, "The  $h - p$  finite element modeling of plates and shells" (M. Suri)
- Mar '95** University of Texas, Austin, TX, "The  $h - p$  finite element modeling of plates and shells" (M. Suri)
- April '94** (Conference on Multiscale Methods in Numerical Analysis organized by the Weierstrass Institute, Berlin) "Fully Discrete Wavelet Galerkin Boundary Element Methods" (C. Schwab)

**Jul '95** ICIAM '95 (International Congress of Industrial and Applied Mathematics), "Mixed  $h - p$  methods for plates and shells" (M.Suri) (PIs co-organized minisymposium entitled "Modeling Thin Structures in Mechanics — Asymptotic and Finite Element Analysis")

**Jul '95** ICIAM '95 (International Congress of Industrial and Applied Mathematics), "Fully Discrete Wavelet Galerkin Boundary Element Methods" (C. Schwab)

## Publications and Preprints

- [1] C. Kim and M. Suri, "On the  $p$  version of the finite element method in the presence of numerical integration", *Numerical Methods for Partial Differential Equations*, 9, 593-629 (1993).
- [2] C. Schwab, "The Fourier Approach to the asymptotic analysis of plate models" Mathematics Research Report 93-4, Dept. of Mathematics, UMBC.
- [3] T. von Petersdorff and C. Schwab, "Wavelet discretization of first kind boundary integral equations on polygons", Technical Note BN-1157, Inst. Phys. Science and Technology, Univ. Maryland, College Park, USA December 1993 (in press in Numerische Mathematik)
- [4] C. Schwab, "A-posteriori error estimation for hierachic plate models", Proceedings of the EUROMECH 302 conference, Paris, May 1993 (P.G. Ciarlet and E. Sanchez-Palencia Eds.)
- [5] I. Babuška and C. Schwab, "A-posteriori error estimation for hierachic models of elliptic boundary value problems on thin domains" (Techn. Note BN-1148, IPST, Univ. of Md, College Park) in press in SIAM Journ. Num. Anal.
- [6] I. Babuška, I. Lee and C. Schwab, "On the a-posteriori estimation of the modelling error for the heat conduction in a plate and its use for adaptive hierachic modelling" Applied Numerical Mathematics **14** (1994) 5-21.
- [7] C. Schwab, "Variable order composite quadrature of singular and nearly singular integrals" Computing **53** 173-194 (1994)

- [8] C. Schwab and M. Suri, "Locking and boundary layer effects in the finite element approximation of the Reissner-Mindlin plate model," *Mathematics of Computation* 1943-1993: A half-century of computational mathematics (W. Gautschi, ed.), Proc. Symposia Appl. Math., vol. 48, Amer. Math. Soc., Providence, R.I., 1994, pp. 367-371.
- [9] I. Babuška and M. Suri, "The  $p$  and  $h-p$  versions of the finite element method: basic principles and properties", *SIAM Review*, 36, 578-632 (1994).
- [10] B. Kawohl and C. Schwab, "Convergent finite elements for a class of nonconvex variational problems," Universitat Heidelberg, Preprint 94-10 (SFB 359), 1994, submitted to *Numer. Math..*
- [11] C. Schwab, "Boundary Layer Resolution in Hierarchic Models of Laminated Composites", *R.A.I.R.O Analyse Numérique*, Sér. rouge. **28** 1994, 517-537
- [12] R. Stenberg and M. Suri, "Mixed  $h - p$  finite element methods for elasticity problems," (to appear in *Numerische Mathematik*, 1995).
- [13] M. Suri, "A reduced constraint  $h - p$  finite element method for shell problems," Technical Report SUR 94-02, Department of Mathematics and Statistics, UMBC, June, 1994 (submitted to *Mathematics of Computation*).
- [14] R. Stenberg and M. Suri, "An  $h - p$  error analysis of MITC plate elements," Technical Report SUR 94-04, Department of Mathematics and Statistics, UMBC, Dec, 1994 (to appear in *SIAM Journal on Numerical Analysis*, April, 1997).
- [15] M. Suri, I. Babuška and C. Schwab, "Locking effects in the finite element approximation of plate models", *Mathematics of Computation*, 64, 461-482 (1995).
- [16] C. Schwab and S. Wright, "Boundary Layers of Hierarchic Beam and P0. Journal of Elasticity **38** (1995), 1-40 (with S. Wright).
- [17] I. Babuška and C. Schwab, "Subspace correction methods for hierarchic plate models I: heat conduction problems", Technical Note BN-1169, IPST Univ. of Maryland College Park, April 1994 "Comp. Meth. Appl. Mech. Engg." 1995, in press.

- [18] M. Suri, "Analytic and computational assessment of locking in the  $h-p$  finite element method," (to appear in *Computer Methods in Applied Mechanics and Engineering*, 1995).
- [19] C. Schwab and M. Suri, "The  $p$  and  $h-p$  versions of the finite element method for problems with boundary layers," Technical Report SUR 95-01, Department of Mathematics and Statistics, UMBC, Jan, 1995 (submitted to *Mathematics of Computation*).
- [20] C. Schwab, M. Suri and C.A. Xenophontos, "Boundary layer approximation by spectral/  $h-p$  methods," to appear in *Houston Journal of Mathematics*.
- [21] C. Schwab, "Mesh design for structural plates," Technical brief, Engineering Software Research and Development, Inc, St. Louis, MO, 1995.
- [22] J. Pitkäranta and M. Suri, "Design principles and error analysis for reduced-shear plate-bending finite elements," Technical Report SUR 95-02, Department of Mathematics and Statistics, UMBC, Jan, 1995 (submitted to *Numerische Mathematik*).
- [23] T. von Petersdorff, R. Schneider and C. Schwab, "Multiscale methods for integral equations on surfaces in  $\mathbb{R}^3$ " (submitted to SINUM).
- [24] C. Schwab, "A-posteriori error estimation for hierachic plate models" (to appear in Numerische Mathematik 1996).
- [25] C. Schwab and M. Suri, "The optimal  $p$  version approximation of singularities on polyhedra in the boundary element method", (to appear in *SIAM Journal on Numerical Analysis*, 1995).